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Ship waves in a stratified ocean

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This paper considers the effect of a pressure point moving over the surface of a fluid which is composed of two layers of different densities, the lower layer being of infinite depth. The method is the same as that used by Crapper (1964) for the same problem with a uniform fluid. The resulting waves are in two sets, the first being the normal surface wave for the uniform ocean, with a slight change in amplitude, and the second an internal wave. The crests of the internal wave have a pattern similar to that of the surface wave for a shallow uniform fluid.

1. Introduction and formal solution

Hudimac (1961) considers the waves generated by a simple source moving at constant speed and constant depth below the surface of a two-layer ocean in which the layers have different densities, and the lower layer is of infinite depth. He points out that in reality the ocean often has this kind of structure. The problem is solved by Fourier analysis, and an asymptotic expansion is found for the velocity potential. However, the method of dealing with the asymptotics is rather cumbersome, and is much simplified in the present paper, which also calculates the effect of the stratification on the 'surface wave mode', i.e. the waves which would be present without the stratification. Hudimac did not give this result.

In order to keep the detail as simple as possible the present paper is restricted to the waves generated by a pressure point moving over the surface. The submerged source disturbance considered by Hudimac is not essentially more difficult, and the wave patterns are identical, but the surface pressure disturbance is simpler and enables easy comparison with Crapper (1964), in which the problem was considered for a uniform ocean. The method we shall use is exactly the same as in that paper, and is due to Lighthill (1960).

We consider a steady situation in which the pressure point is fixed at the origin and the flow far upstream has velocity U in the x-direction. When undisturbed, the upper layer of fluid, which has density ρ, occupies the region $-h < z \leqslant 0$ and the lower layer, which has density $\rho(1+s)$, occupies the region $-\infty < z < -h$. In each layer the flow is irrotational, and the velocity potentials are

$$Ux+\phi_1, \quad Ux+\phi_2; \tag{1}$$

ϕ_1 and ϕ_2 are assumed small so that the boundary conditions can be linearized. The continuity equation gives

$$\nabla^2\phi_1 = \nabla^2\phi_2 = 0. \tag{2}$$

The linearized surface boundary condition, from Bernoulli's equation, neglecting surface tension, is

$$P\delta(x)\delta(y)+\tfrac{1}{2}\rho\left(U^2+2U\frac{\partial\phi_1}{\partial x}\right)+\rho g\eta = \text{const.} = \tfrac{1}{2}\rho U^2 \quad \text{on} \quad z=0, \tag{3}$$

where P is the magnitude of the applied pressure; $\delta(\ )$ is the Dirac delta-function; g is the acceleration due to gravity; and $\eta(x,y,z)$ is the elevation above its initial level of the streamline which when undisturbed was at height z. We also have the kinematic condition

$$\frac{d\eta}{dt} = U\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial z}, \tag{4}$$

where $\phi = \phi_1$ or ϕ_2 according as we are in the upper or lower layer. At the interface η has to be the same in each layer, i.e.

$$\frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z} \quad \text{at} \quad z=-h, \tag{5}$$

and the pressure has to be continuous, which gives, from the linearized Bernoulli equation,

$$\rho\left(U\frac{\partial\phi_1}{\partial x}+g\eta\right) = \rho(1+s)\left(U\frac{\partial\phi_2}{\partial x}+g\eta\right) \quad \text{at} \quad z=-h. \tag{6}$$

Finally,

$$\phi_2 \to 0 \quad \text{as} \quad z\to-\infty. \tag{7}$$

We now adopt non-dimensional variables, taking U^2/g as unit of length, U as unit of velocity and ρ as unit of density. We can then simply put $U=g=\rho=1$ in equations (1) to (7), and η, ϕ_1, etc., are now non-dimensional.

Defining Fourier transforms by

$$\phi(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[i(\alpha x+\beta y)\right]\phi^{(0)}(\alpha,\beta,z)\,d\alpha\, d\beta, \tag{8}$$

we take transforms of (2), giving

$$\frac{\partial^2\phi^{(0)}}{\partial z^2}-(\alpha^2+\beta^2)\,\phi^{(0)} = 0; \quad \phi^{(0)} = \phi_1^{(0)}, \quad \text{or} \quad \phi_2^{(0)}. \tag{9}$$

Thus

$$\phi_1^{(0)} = A(\alpha,\beta)\,e^{kz}+B(\alpha,\beta)\,e^{-kz} \tag{10}$$

and

$$\phi_2^{(0)} = C(\alpha,\beta)\,e^{kz}, \tag{11}$$

where

$$k = \sqrt{(\alpha^2+\beta^2)} > 0, \tag{12}$$

and we have eliminated a possible second term in the right-hand side of (11) by using (7). The functions A, B and C can now be determined from the transforms of (3) to (6). Then taking the inverse transform we find

$$\eta = \frac{P}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left[i(\alpha x+\beta y)\right]F(\alpha,\beta)\,d\alpha\, d\beta}{G_S(\alpha,\beta)\,G_I(\alpha,\beta)} \quad (-h< z\leqslant 0); \tag{13}$$

$$= -\frac{P}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left[i(\alpha x+\beta y)\right]k\alpha^2\exp\left[k(z+h)\right]d\alpha\, d\beta}{G_S(\alpha,\beta)\,G_I(\alpha,\beta)} \quad (-\infty< z< -h); \tag{14}$$

where

$$F(\alpha,\beta) \equiv k[sk\sinh k(z+h) - \alpha^2\{(1+s)\sinh k(z+h) + \cosh k(z+h)\}]; \quad (15)$$

$$G_S(\alpha,\beta) \equiv \alpha^2 - k; \quad (16)$$

$$G_I(\alpha,\beta) \equiv [sk\sinh kh - \alpha^2\{(1+s)\cosh kh + \sinh kh\}]. \quad (17)$$

As $h \to \infty$ in (13) or $h \to 0$ in (14) we recover the integral for a uniform ocean (Crapper 1964, equation (9) with $h \to \infty$, $\sigma = 0$). In addition, we note that the two integrals are equal at the interface $z = -h$.

2. Asymptotic expansions

We now use the method of Crapper (1964), following that of Lighthill (1960). As the method is given in detail in the earlier paper it will only be outlined here. We first choose new independent variables $\bar{\alpha}$, $\bar{\beta}$ such that

$$\alpha = \bar{\alpha}\cos\theta - \bar{\beta}\sin\theta, \quad \beta = \bar{\alpha}\sin\theta + \bar{\beta}\cos\theta, \quad (18)$$

where $x = r\cos\theta$, $y = r\sin\theta$ for the particular $\mathbf{r} = (x,y)$ which we consider. We then evaluate the integral by contour integration, the main contribution coming from poles where either G_S or G_I vanishes. The poles are on the real $\bar{\alpha}$ axis, and to avoid the non-uniqueness of the result which this presents we would normally use a radiation condition. Here, however, we shall simply assume that the waves appear downstream of the disturbance, i.e. for $\cos\theta > 0$. The results of the earlier paper, together with those of Hudimac, confirm that this is the actual result. Thus we have

$$\eta \sim \frac{P}{4\pi^2}\int_{-\infty}^{\infty} 2\pi i\left\{\Sigma\frac{\exp(i\bar{\alpha}r)F}{(\partial G_S/\partial\bar{\alpha})G_I} + \Sigma\frac{\exp(i\bar{\alpha}r)F}{G_S(\partial G_I/\partial\bar{\alpha})}\right\}d\bar{\beta} \quad (-h < z \leqslant 0), \quad (19)$$

where the summations are over parts of the lines $G_S = 0$ and $G_I = 0$ respectively, the actual parts depending on the precise radiation condition. There is a corresponding result for the lower layer.

Now the $\bar{\beta}$ integral is evaluated by the method of stationary phase. It turns out that the stationary points are where the normals to the lines $G_S = 0$ and $G_I = 0$ are parallel to $\mathbf{r}$, for the particular x, y chosen; they only count for $\cos\theta > 0$ because of the radiation condition. Since the poles in the $\bar{\alpha}$ integral give rise to waves, this simple geometrical condition gives the vector wave-number for given x, y and makes it easy to draw the wave-crests. If there are no normals in a given direction, then there are no waves in that direction. The wave-crests depend on the nature of the singularity lines $G_S = 0$ and $G_I = 0$. With no stratification ($s = 0$) the only singularities are where $G_S = 0$. Hence this line describes a surface wave mode. We shall see that $G_I = 0$ describes an internal wave mode. The final result can be written

$$\eta \sim \frac{P_i}{2\pi}\left\{\sum_m \frac{F}{(\partial G_S/\partial\bar{\alpha})G_I}\sqrt{\left(\frac{2\pi}{|\kappa_S|\,r}\right)}\exp\left(i\bar{\alpha}_m r + \tfrac{1}{4}\pi i\,\mathrm{sgn}\,\kappa_S\right)\right.$$

$$\left. + \sum_m \frac{F}{G_I(\partial G_S/\partial\bar{\alpha})}\sqrt{\left(\frac{2\pi}{|\kappa_I|\,r}\right)}\exp\left(i\bar{\alpha}_m r + \tfrac{1}{4}\pi i\,\mathrm{sgn}\,\kappa_I\right)\right\} + O(r^{-1}), \quad (20)$$

where K_S, K_I are the curvatures of the singularity lines; $\operatorname{sgn}\kappa = \pm 1$ according as $\kappa \gtrless 0$; and the summation is over the stationary points $(\bar{\alpha}_m, \bar{\beta}_m)$ for which $\cos\theta > 0$. Because of the symmetry of the curves $G_S = 0$ and $G_I = 0$ these appear in pairs, and a real result is obtained.

3. Results

The curve $G_S = 0$ is shown in figure 1; it is exactly the same curve as in the case of a uniform ocean of infinite depth. Thus it leads to the familiar pattern of waves on the surface. The only difference from the uniform ocean case is a factor in the amplitude. Here one factor in the residue at the pole $G_S = 0$ is

$$\left(\frac{F}{G_I}\right)_{\alpha^2=k} = \frac{k\mathrm{e}^{kz}}{1+s\,e^{-2kh}}, \tag{21}$$

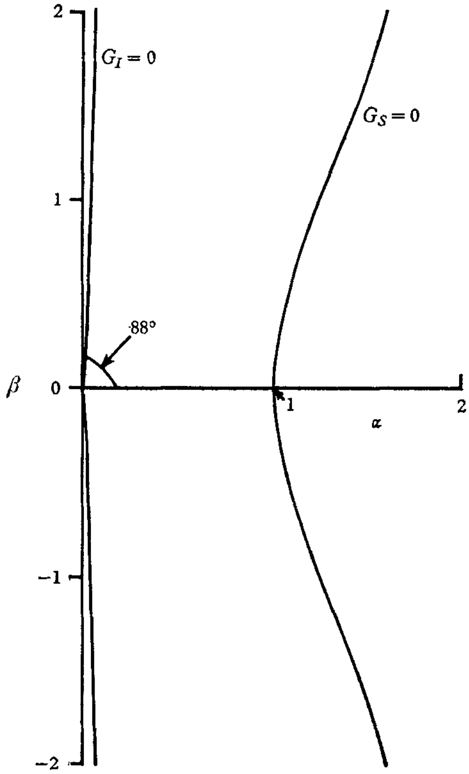

FIGURE 1. Singularity lines $G_S = 0$ and $G_I = 0$ (for $h = 1$, $s = 0{\cdot}001$ ($\mathscr{F} \sim 32$)). The lines are symmetrical about both axes and only the right-hand half is shown.

and in the uniform ocean case this is replaced simply by ke^{kz}. The other factors in the first term of (20) are equal in the two cases. Thus the surface wave mode has its amplitude changed by a factor $(1+s\,e^{-2kh})^{-1}$ in the presence of stratification, and its amplitude still decays exponentially with depth. Because of the k in this factor the difference of amplitude changes across the wave pattern, since k takes the value for the appropriate stationary point, being maximum on the x-axis where k is least (the normal to $G_S = 0$ parallel to $(x, 0)$ is at the point

(1, 0) in the (α, β)-plane (see figure 1), and this is therefore the stationary point for waves on the x-axis.)

The curvature K_S actually vanishes at points with normals parallel to $\mathbf{r}$ when $|\theta| = 19° 28'$, along the edges of the wedge in which the waves are contained. The first term in (20) is then not correct (and it is not very good where κ_S is small) and the correct version which shows a decay of amplitude proportional to $r^{-\frac{1}{3}}$, is Crapper (1964, equation (38)), with the amplitude multiplied by

$$(1 + s\,e^{-2kh})^{-1}$$

and the appropriate value of $k(k = \frac{3}{2})$.

The other mode depends on the shape of the curve $G_I = 0$, which in turn depends on both s and h. This equation is best written in the form

$$\alpha^2 = \frac{sk \tanh kh}{1 + s + \tanh kh}. \tag{22}$$

For $\mathscr{F}^2 = (1+s)/sh > 1$ (Hudimac (1961) calls this parameter F^2, not to be confused with our $F(\alpha, \beta)$) the curve is similar to that shown for particular values in figure 1, while for $\mathscr{F}^2 < 1$ the shape is similar to that of the curve $G_S = 0$, but cutting the α-axis nearer to the origin. Hudimac calculates amplitudes, etc., for the values $\mathscr{F} = \frac{1}{2}$, 2 and in some cases up to $\mathscr{F} = 10$, with $s = 0{\cdot}001$; these correspond to non-dimensional depths $h \sim 4000$, 250 and 10 respectively. The results for a uniform ocean of non-dimensional finite depth h show that for $h \geqslant 2$ there is no appreciable change from the infinite depth solution. Any density stratification with a depth $h \geqslant 2$ is therefore not going to make any change from the uniform ocean results for the present problem. With $s = 0{\cdot}001$ and $h = 2$ we have $\mathscr{F} \sim 22$, and to make $\mathscr{F} = 1$ with $h = 2$ we need $s = 1$, i.e. the lower layer with density twice that of the upper layer—a situation not likely to arise in practice. Hudimac's calculations therefore do not have much value for the pressure point problem. However, as pointed out above, he actually considers the waves generated by a submerged source, which in his calculations he takes to be near to the interface between the layers. Thus in his case, with large values of h, the surface wave mode is likely to be of no importance, but small values of $\mathscr{F}$, which essentially means a slowly moving source, may be important. Since when $\mathscr{F} < 1$ the curve $G_I = 0$ is similar to the curve $G_S = 0$, the predicted wave patterns will clearly be similar to those of the surface wave mode, although with different (larger) wedge angles and longer wavelength, rather like surface waves on a uniform ocean of not too small finite depth (Crapper (1964) $1 < h < 2$).

However, we can concentrate on $\mathscr{F} > 1$. As $k \to \infty$ the curve (22) behaves like a parabola $\alpha^2 = s\beta/(2+s)$, so that for small s the curve is very close to the β-axis. If we put $\alpha = k \cos \chi$ we find that as $k \to 0$, $\chi \to \cos^{-1}(\mathscr{F}^{-1})$. Thus the normals to the curve make angles between zero and $\pm\theta_c = \pm(\frac{1}{2}\pi - \cos^{-1}(\mathscr{F}^{-1}))$ with the x-axis, and the waves appear only within the wedge $|\theta| < \theta_c$. The wave patterns are similar to those given by Hudimac (1961, figure 7) and, indeed, are like the surface wave crests for a shallow uniform ocean. They correspond to the diverging wave system of the surface wave mode. For fixed h, $\theta_c = O(s^{\frac{1}{2}})$ and will in practice be a small angle, much smaller than the corresponding angle for the surface wave mode.

In considering the amplitude of the internal wave mode there are three features which we wish to investigate. The first is the behaviour with depth, which in the upper layer is given by

$$|F(\alpha,\beta)|_{G_I=0} \tag{23}$$

since z does not appear in the other factors. To evaluate this factor, substitute for α^2 from (22) into (15);

$$|F(\alpha,\beta)|_{G_I=0} = \frac{|sk^2[\sinh k(z+h)\{(1+s)\cosh kh+\sinh kh\} - \sinh kh\{(1+s)\sinh k(z+h)+\cosh k(z+h)\}]|}{|(1+s)\cosh kh+\sinh kh|}, \tag{24}$$

$$= \left|\frac{sk^2[\sinh kz + s\,e^{-kh}\sinh k(z+h)]}{(1+s)\cosh kh+\sinh kh}\right| \tag{25}$$

after some manipulation. As depth increases ($z < 0$) the first term in the numerator increases rapidly, while the second has a maximum at the surface and falls off to zero at the interface $z = -h$. For small s, however, the first term dominates, and the amplitude increases rapidly with depth down to the interface, beyond which (25) is no longer part of the solution. Below the interface, (14) shows that the amplitude falls off exponentially with depth. Thus this part of the solution really is an internal wave.

The second feature is the behaviour of the amplitude as a function of s, for fixed h. To consider this in full is very complicated, and here we shall be content with finding the behaviour for small s. We note that from (22), regarding k as $O(1)$, $\alpha^2 = O(s)$ and therefore $\beta = 0(1)$. Then from equations (15), (16) and (17), $F = O(s)$, $G_S = O(1)$ and $G_I = O(s)$. Crapper (1964, equations (24), (25)) shows that

$$\frac{\partial G_I}{\partial \bar{\alpha}} = \frac{1}{\cos\theta}\frac{\partial G_I}{\partial \alpha} \tag{26}$$

and

$$\kappa_I = -\frac{\partial^2 G_I}{\partial \bar{\beta}^2}\bigg/\frac{\partial G}{\partial \bar{\alpha}} \tag{27}$$

and here, since $\theta = O(s^{\frac{1}{2}})$, $\cos\theta = O(1)$ and $\sin\theta = O(s^{\frac{1}{2}})$. It follows that

$$\partial/\partial\bar{\alpha} \sim \partial/\partial\alpha = O(s^{-\frac{1}{2}}) \quad \text{and} \quad \partial/\partial\bar{\beta} \sim \partial/\partial\bar{\beta} = O(1),$$

and hence $\partial G_I/\partial\bar{\alpha} = O(s^{\frac{1}{2}})$ and $\kappa_I = O(s^{\frac{1}{2}})$ giving a contribution to η from the internal wave of $O(s^{\frac{1}{4}})$. We conclude that the amplitude of the internal waves tends to zero only slowly as the density difference disappears, and we have the possibility of large internal waves for small differences of density. Thirdly, if s is fixed and we let h become small, similar arguments, with now $\theta = O(h^{\frac{1}{2}})$ and $z = O(h)$ in the upper layer, show that the internal wave amplitude is $O(h^{\frac{1}{4}})$, and shallow layers may give large internal waves.

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